fields. This, in turn, leads one to expect that the probability $p(r)$ of counting r photons in δV will be given by a Poisson distribution with parameter U, which is then to be averaged over the ensemble of U. Thus,

$$p(r)=\left\langle\left\langle\frac{U^r}{r!}\exp(-U)\right\rangle\right\rangle. \tag{20}$$

Further details of the argument leading to (20) are given in Refs. 10 and 11.

It will now be seen that the nth moment of r, i.e., of the counts, corresponds to the K_n defined quantum mechanically by (10), whereas the nth moment of U, i.e., of the classical integrated intensity, corresponds to the quantum correlation L_n given by (11). The moment-generating function for r is given by

$$M_r(y)=\sum_{r=0}^{\infty}\exp(ry)p(r);$$

and from (20), and with the help of the well-known properties of the Poisson distribution, we arrive at

$$\begin{aligned}M_r(y)&=\langle\langle\exp[U(e^y-1)]\rangle\rangle\\&=M_U(e^y-1),\end{aligned} \tag{21}$$

by definition of the moment-generating function for U. This relation is the semiclassical equivalent of the quantum-mechanical equation (17).

The result illustrates once again that normal-ordered operators correspond to correlations of the complex field in the semiclassical treatment. As the relations (17) and (21) hold for any state of the field, we see that the semiclassical theory may sometimes be just as accurate as the quantized field theory, while providing some valuable intuitive insight into the physics of the problem.

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Gravitational Radiation and the Motion of Two Point Masses

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The expansion of the field equations of general relativity in powers of the gravitational coupling constant yields conservation laws of energy, momentum, and angular momentum. From these, the loss of energy and angular momentum of a system due to the radiation of gravitational waves is found. Two techniques, radiation reaction and flux across a large sphere, are used in these calculations and are shown to be in agreement over a time average. In the nonrelativistic limit, the energy and angular momentum radiation and angular distributions are expressed in terms of time derivatives of the quadrupole tensor Q_{ij}. These results are then applied to a bound system of two point masses moving in elliptical orbits. The secular decays of the semimajor axis and eccentricity are found as functions of time, and are integrated to specify the decay by gravitational radiation of such systems as functions of their initial conditions.

I. INTRODUCTION

THE existence of gravitational radiation was predicted by Einstein[1,2] shortly after he formulated his general theory of relativity. Systems of moving masses should emit gravitational waves in analogy with the emission of electromagnetic waves by a system of moving charges. Early attempts to calculate the energy in these waves were based on the use of a pseudostress-energy tensor for the evaluation of the energy flux. One disadvantage of this method is that one can always choose a coordinate system in which the energy flux vanishes.[3] This led to much scepticism about the reality of gravitational radiation. Another disadvantage of the calculation is that it is valid only for systems which are not gravitationally bound. Thus, the important case of gravitational radiation from binary stars remained unsolved at that time.

Later, Eddington found the radiation from a system by calculating the radiation reaction of the system on itself.[4] However, like Einstein's method, this is not valid for gravitationally bound systems. For situations in which the radiation is constant, the two methods agree; for situations in which the radiation is time-dependent, the answers differ. One can show that over a time average of the motion the two answers are in agreement. Analogous results occur in the theory of electromagnetic radiation.

For systems in which the velocities of the masses are small compared to the velocity of light, the calculation of Einstein has been extended to include gravitationally

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[1] A. Einstein, Sb. Preuss. Akad. Wiss. 688 (1916).
[2] A. Einstein, Sb. Preuss. Akad. Wiss. 154 (1918).
[3] For a detailed discussion of the status of the theories of gravitational radiation and their objections, the reader is referred to the review article by F. A. E. Pirani, in *Gravitation: An Introduction to Current Research*, edited by L. Witten (John Wiley & Sons, Inc., New York, 1962), Chap. 6.
[4] A. S. Eddington, Proc. Roy. Soc. (London) **A102**, 268 (1922).

bound systems.[5] The problem concerning the choice of the stress-energy of the gravitational field is still debated. Also, the selection of certain preferred coordinate systems and conditions is subject to much criticism. One can find references in the current literature which describe the radiation from the system as carrying away energy,[5] bringing in energy,[6] carrying no energy,[7] or having an energy dependent on the coordinate system used.[7] Clearly, a consistent picture of gravitational radiation is desirable.

One approach to gravitational radiation is to consider only exact solutions of the nonlinear field equations of general relativity. All such solutions found so far correspond to unphysical systems.[8] Therefore, one usually employs some approximation procedure in solving the field equations. The field equations are sometimes expanded in powers of the gravitational coupling constant because of the weakness of the gravitational interaction. In addition, one encounters expansions in powers of the ratio of the velocities of the masses of the system to the velocity of light, and also expansions in inverse powers of distance from the system under consideration. These approximation methods are not independent. Throughout this paper we shall be concerned only with solutions obtained using these expansions, and not with exact solutions of the field equations. We shall, however, keep all terms of the expansions until they are clearly negligible in the approximation in which we will be working.

In Sec. II, the field equations of general relativity are expanded in powers of the gravitational coupling constant, and from these, integral conservation laws of energy, momentum, and angular momentum are obtained. In Sec. III, these results are used to find the energy loss of a system radiating gravitational waves. Two methods, radiation reaction and energy flux across a large sphere, are used in finding the energy radiated by an arbitrary system. For a nonrelativistic system, the radiation is given in terms of time derivatives of the matter distribution of the system. In Sec. IV, the angular momentum loss of a system is found by methods analogous to those used in the energy loss case. In the nonrelativistic limit, the angular momentum loss can be simplified in the same manner as the energy loss. Section V treats the system of two point masses moving in elliptical orbits under their mutual gravitational attraction. The previous analysis is used to show that the system must decay as a result of gravitational radiation and that the changes in the elements of the relative orbit can be found during such a decay.

[5] See, for example, L. Landau and E. Lifshitz, *The Classical Theory of Fields* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1959), Chap. 11.

[6] P. Havas and J. N. Goldberg, Phys. Rev. **128**, 398 (1962).

[7] L. Infeld and J. Plebanski, *Motion and Relativity* (Pergamon Press, Inc., New York, 1960), Chap. VI.

[8] For an example, see J. Weber, *General Relativity and Gravitational Waves* (Interscience Publishers, London, 1961), pp. 99–105.

II. CONSERVATION LAWS

We shall assume that the field equations of Einstein are valid[9]:

$$R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R=-8\pi GT_{\mu\nu}. \tag{2.1}$$

Letting $g_{\mu\nu}=\delta_{\mu\nu}+h_{\mu\nu}$, we can expand the field equations in powers of $h_{\mu\nu}$ to get

$$\bar{h}_{\mu\nu,\lambda\lambda}-\bar{h}_{\mu\lambda,\lambda\nu}-\bar{h}_{\nu\lambda,\lambda\mu}+\delta_{\mu\nu}\bar{h}_{\lambda\sigma,\lambda\sigma}=-16\pi GS_{\mu\nu}, \tag{2.2}$$

where

$$\bar{h}_{\mu\nu}=h_{\mu\nu}-\tfrac{1}{2}\delta_{\mu\nu}h_{\sigma\sigma}.$$

$S_{\mu\nu}$ is a combination of the matter tensor $T_{\mu\nu}$ and all of the nonlinear terms containing the $h_{\mu\nu}$.

$$S_{\mu\nu}=T_{\mu\nu}+\sum_{k=2}^{\infty}X_{\mu\nu}{}^{(k)}, \tag{2.3}$$

where $X_{\mu\nu}{}^{(k)}$ is an expression involving the product of k $h_{\mu\nu}$'s and their derivatives. This $S_{\mu\nu}$ is uniquely defined by the field equations. The ordinary divergence of the left side of (2.2) vanishes. Thus, we can conclude that

$$S_{\mu\nu,\nu}=0, \tag{2.4}$$

and hence we can write integral conservation laws for the quantities $\int S_{44}dV$, $\int S_{4i}dV$, and $\int e_{ijk}x_jS_{4k}dV$:

$$\frac{d}{dt}\int S_{44}dV=\int S_{4i}dS_i, \tag{2.5}$$

$$\frac{d}{dt}\int S_{4i}dV=\int S_{ij}dS_j, \tag{2.6}$$

$$\frac{d}{dt}\epsilon_{ijk}\int x_jS_{4k}dV=\epsilon_{ijk}\int x_jS_{kl}dS_l. \tag{2.7}$$

On the right side of Eqs. (2.5) to (2.7), the volume integrals have been converted to surface integrals.

In an approximation in which there is no matter or radiation entering or leaving the system, the right sides of Eqs. (2.5) to (2.7) vanish, and thus the integrals on the left-hand sides of the equations must be conserved quantities. In the approximation in which the velocities of the masses are small compared to the velocity of light, it is easy to show that, regardless of coordinate conditions chosen, $\int S_{44}dV$ reduces to the usual expression for the energy of the system, even in the case of gravitationally bound systems. Likewise, $\int S_{4i}dV$ reduces to the negative of the ith component of the momentum of

[9] Greek letters take values from 1 to 4; Latin letters are restricted to spatial components 1 to 3. The Kronecker delta $\delta_{\mu\nu}$ is defined by $\delta_{44}=1$, $\delta_{11}=\delta_{22}=\delta_{33}=-1$, and $\delta_{\mu\nu}=0$ for $\mu\neq\nu$. The summation convention $a_\mu b_\mu=a_4b_4-a_ib_i$ is employed, where $a_ib_i=\mathbf{a}\cdot\mathbf{b}$. Ordinary differentiation is denoted by a comma: $\partial/\partial x_\alpha=,\alpha$. Throughout most of the paper we shall set $c=1$. The antisymmetric tensor ϵ_{ijk} is defined so that $\epsilon_{ijk}=1$ if i, j, $k=1$, 2, 3 or a cyclic permutation thereof, -1 if i, j, $k=1$, 3, 2 or a cyclic permutation thereof, and 0 if any two indices are equal.

the system, and $\int \epsilon_{ijk} x_j S_{4k} dV$ reduces to the negative of the ith component of the angular momentum of the system. Thus, defining E, P_i, and L_i to be the energy, momentum, and angular momentum of the system respectively, we get

$$\frac{dE}{dt} = \int S_{4i} dS_i, \tag{2.8}$$

$$\frac{dP_i}{dt} = -\int S_{ij} dS_j, \tag{2.9}$$

$$\frac{dL_i}{dt} = -\epsilon_{ijk} \int x_j S_{kl} dS_l. \tag{2.10}$$

The choice of coordinate conditions or specific coordinate systems in which calculations are carried out is a source of some confusion in the study of gravitational radiation. The invariance of (2.1) under arbitrary coordinate transformations implies that one may always choose a coordinate system in which $\bar{h}_{\mu\nu,\nu} = 0$. Although this simplifies the work greatly [e.g., Eq. (2.2) then becomes an ordinary inhomogeneous wave equation], it is not an essential restriction on the calculation of the radiation. If one assumes that the expanded field equations are valid for large distances from the system, and that the gravitational potentials $h_{\mu\nu}$ are inversely proportional to the distance from the system for large distances, then the energy of the system must decrease as a result of the radiation of gravitational waves, regardless of coordinate system or conditions used.[10] In the nonrelativistic approximation, the radiation is the same as that found in the gauge $\bar{h}_{\mu\nu,\nu} = 0$. Thus, in the rest of this paper, we shall set $\bar{h}_{\mu\nu,\nu} = 0$.

III. ENERGY LOSS

Equation (2.8) gives us the energy loss from a system once we have S_{4i} from the expansion of the field equations. The surface of integration will be taken to be a large sphere enclosing the system. Since the potentials $h_{\mu\nu}$ and their derivatives will be asymptotically proportional to $1/r$, only those terms of the gravitational field stresses which are products of two potentials or their derivatives will count in (2.8). The terms in the surface integral over $X_{4i}^{(k)}$ for $k \geq 3$ will yield a contribution which falls off like $1/r$ or faster as $r \to \infty$. Thus, (2.8) can be written

$$\frac{dE}{dt} = \int_S X_{4i}^{(2)} dS_i. \tag{3.1}$$

The second-order gravitational stress energy can be determined easily from the expansion of (2.1). We can eliminate all terms which are proportional to $\bar{h}_{\alpha\beta,\beta}$ since this has been chosen to be zero. Also, any terms proportional to $\bar{h}_{\alpha\beta,\lambda\lambda}$ will yield terms in (3.1) proportional to $1/r$, which can be neglected for large r, since $\bar{h}_{\alpha\beta,\lambda\lambda} \propto X_{\alpha\beta}^{(2)} \propto 1/r^2$ as $r \to \infty$. The remaining terms give

$$\frac{dE}{dt} = -\frac{1}{32\pi G} \int_S [h_{\alpha\beta,4} h_{\alpha\beta,i} + 2h_{\alpha 4,\beta} h_{\alpha i,\beta} - 2h_{4\alpha,\beta} h_{i\beta,\alpha} + 2h_{4i,\alpha\beta} h_{\alpha\beta} + 2h_{\alpha\beta} h_{\alpha\beta,4i} - 2h_{\alpha\beta} h_{4\alpha,\beta i} - 2h_{\alpha\beta} h_{i\alpha,\beta 4}] dS_i. \tag{3.2}$$

In the gauge $\bar{h}_{\mu\nu,\nu} = 0$, Eq. (2.2) reduces to

$$\bar{h}_{\alpha\beta,\lambda\lambda} \equiv \Box \bar{h}_{\alpha\beta} = -16\pi G S_{\alpha\beta}. \tag{3.3}$$

This has the well-known retarded solution

$$\bar{h}_{\alpha\beta}(\mathbf{r},t) = -4G \int \left[\frac{S_{\alpha\beta}(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}\right]_{t-|\mathbf{r}-\mathbf{r}'|} dV', \tag{3.4}$$

where the brackets indicate that the quantity within is to be evaluated at the retarded time. For r large compared to the dimensions of the system, this becomes

$$\bar{h}_{\alpha\beta}(\mathbf{r},t) = -\frac{4G}{r} \int [S_{\alpha\beta}(\mathbf{r}',t)]_{t-|\mathbf{r}-\mathbf{r}'|} dV'. \tag{3.5}$$

If we neglect the parts of $\bar{h}_{\alpha\beta}$ and its derivatives which are proportional to $1/r^2$ and higher, then we can write

$$\bar{h}_{\alpha\beta,i} = \frac{\partial}{\partial x_i} \bar{h}_{\alpha\beta} = -\frac{x_i}{r} \bar{h}_{\alpha\beta,4} \equiv -n_i \bar{h}_{\alpha\beta,4}. \tag{3.6}$$

Thus, in Eq. (3.2), all of the spatial derivatives which appear can be converted to time derivatives multiplied by appropriate direction vectors n_i.

Consider the time average of Eq. (3.2). We shall assume that, for periodic motion, the secular change in the parameters describing the system can be neglected over one period of the motion. This is in analogy with the theory of electromagnetic radiation where two methods of finding the energy loss, radiation reaction and energy flux across a large surface, agree only over a time average. Thus, any terms in the reduced $X_{4i}^{(2)}$ which are pure time derivatives can be then converted to secular changes, which are assumed to be negligible in this approximation. Since the terms in the surface integral of (3.2) can be reduced to a form where only time derivatives are present through the use of (3.6), taking the time average of (3.2) allows us to integrate by parts any derivatives we may choose. For example, letting $n_\alpha = (-1, n_i)$, we get

$$\int dt h_{4i,\alpha\beta} h_{\alpha\beta} = \int dt n_\alpha n_\beta h_{4i,44} h_{\alpha\beta} = \int dt h_{4i} h_{\alpha\beta,44} n_\alpha n_\beta = \int dt h_{4i} h_{\alpha\beta,\alpha\beta},$$

[10] P. C. Peters, Bull. Am. Phys. Soc. 8, 615 (1963).

which can be further reduced through the use of the coordinate condition $\bar{h}_{\mu\nu,\nu}=0$ to $\frac{1}{2}\int dth_{4i}h_{\sigma\sigma,\alpha\alpha}\propto 1/r^3$. This term will not contribute to the surface integral in (3.2) and can be neglected. Simplifying the remaining terms gives

$$\int \frac{dE}{dt}dt=\frac{1}{32\pi G}\int\int [h_{\alpha\beta,4}\bar{h}_{\alpha\beta,i}]dS_i dt \qquad (3.7)$$

or

$$\int \frac{dE}{dt}dt=-\frac{1}{32\pi G}\int\int_{\text{sphere}} [h_{\alpha\beta,4}\bar{h}_{\alpha\beta,4}]dSdt, \qquad (3.8)$$

since $dS_i=n_i dS$ if dS is the differential surface area of a sphere.

There are two directions which we may now take to obtain the energy loss in terms of the matter distribution. We may convert Eq. (3.7) back to an integral over volume in order to find the radiation reaction energy loss, or we may evaluate the surface integral in (3.8) explicitly to get the energy flux across a large sphere. Taking the first alternative, we let $\int dS_i\rightarrow \int dV(\partial/\partial x_i)$. Again eliminating terms which vanish on a time average, we get

$$\int \frac{dE}{dt}dt=\frac{1}{32\pi G}\int\int h_{\alpha\beta,4}\bar{h}_{\alpha\beta,ii}dVdt$$

$$=-\frac{1}{32\pi G}\int\int h_{\alpha\beta,4}\bar{h}_{\alpha\beta,\lambda\lambda}dVdt$$

$$=\frac{1}{2}\int dt\int dVh_{\alpha\beta,4}S_{\alpha\beta}. \qquad (3.9)$$

The quantity $\frac{1}{2}\int dVh_{\alpha\beta,4}S_{\alpha\beta}$ will be called the energy loss by gravitational radiation reaction.[11]

In the case of systems where the velocities of the masses are small compared to the velocity of light, and where retardation effects are also small, we may derive from (3.9) the usual formula for the energy loss in terms of time derivatives of the mass tensor Q_{ij}, where

$$Q_{ij}=\sum_a m^a x_a{}^i x_a{}^j. \qquad (3.10)$$

To do this we consider Eq. (3.4), and use the expansion of the retarded quantities in a Taylor series about the present time.[12] Substituting this expression in (3.9) gives

$$\int \frac{dE}{dt}dt=2G\int\int\int dVdV'dt\Bigg\{S_{\alpha\beta}\frac{d^2S_{\alpha\beta}'}{dt^2}+\tfrac{1}{6}S_{\alpha\beta}|\mathbf{r}-\mathbf{r}'|^2\times\frac{d^4S_{\alpha\beta}'}{dt^4}+\frac{1}{120}S_{\alpha\beta}|\mathbf{r}-\mathbf{r}'|^4\frac{d^6S_{\alpha\beta}'}{dt^6}-\tfrac{1}{2}S_{\alpha\alpha}\frac{d^2S_{\beta\beta}'}{dt^2}-\frac{1}{12}S_{\alpha\alpha}|\mathbf{r}-\mathbf{r}'|^2\frac{d^4S_{\beta\beta}'}{dt^4}-\frac{1}{240}S_{\alpha\alpha}|\mathbf{r}-\mathbf{r}'|^4\frac{d^6S_{\beta\beta}'}{dt^6}\Bigg\}, \qquad (3.11)$$

where all terms which explicitly vanish over a time average have been dropped, and where all of the terms which contribute to the radiation in higher than fifth order in v/c have been neglected. Consider, for example, the first term on the right side of (3.11). There is no contribution if either α or β is 4, since then we could use the conservation law $S_{\alpha\beta,\beta}=0$ to reduce the term to a surface integral which would vanish in the order in which we are finding the energy loss. Thus, making use of the time average, we get a contribution of $-2G\int\dot{S}_{ij}dV\int\dot{S}_{ij}'dV'$. The integral of S_{ij} over space is simply related to the mass tensor Q_{ij}:

$$\int S_{ij}dV=\frac{1}{2}\frac{d^2Q_{ij}}{dt^2}. \qquad (3.12)$$

In the terms of (3.11) where the r's are present, the conservation laws and integrations by parts of spatial and time derivatives are used to transform the terms to the form of a product of two integrals. Carrying out these operations and expressing the result in terms of the Q_{ij}, we get

$$\int \frac{dE}{dt}dt=-\frac{G}{5}\int dt\left[\frac{d^3Q_{ij}}{dt^3}\frac{d^3Q_{ij}}{dt^3}-\frac{1}{3}\frac{d^3Q_{ii}}{dt^3}\frac{d^3Q_{jj}}{dt^3}\right], \qquad (3.13)$$

which agrees, of course, with the usual treatments of gravitational radiation.[5]

The energy loss may be found also from a calculation of the energy flux through a large sphere surrounding the system. We first convert (3.8) to an expression containing only $\bar{h}_{ij,4}$ in the following manner:

$$\int \frac{dE}{dt}dt=-\frac{1}{32\pi G}\int dt\int_{\text{sphere}} dS[\tfrac{1}{2}\bar{h}_{44,4}\bar{h}_{44,4}-2\bar{h}_{4i,4}\bar{h}_{4i,4}+\bar{h}_{ij,4}\bar{h}_{ij,4}+\bar{h}_{ii,4}\bar{h}_{44,4}-\tfrac{1}{2}\bar{h}_{ii,4}\bar{h}_{jj,4}]. \qquad (3.14)$$

[11] An expression similar to (3.9) has been obtained by A. Peres, Phys. Rev. **128**, 2471 (1962). However, his method of finding the energy loss from the covariant divergence of $T^{\mu\nu}$ is not applicable to gravitationally bound systems. In his derivation he replaces the matter tensor $T^{\mu\nu}$ by the total stress-energy tensor $S^{\mu\nu}$. For gravitationally bound systems, the gravitational field stresses X_{ij} are of the same order of magnitude as the matter terms T_{ij}, and both contribute to the radiation in the same order in v/c. This, of course, implies that the covariant divergence condition is not sufficient to determine the radiation in general. The source of gravitational radiation is all stress energy, including that of the binding fields. A consideration of the covariant divergence of the matter tensor $T^{\mu\nu}$ or, equivalently, of the equations of motion, does not give the contribution of the gravitational field stresses.

[12] A. S. Eddington, *The Mathematical Theory of Relativity* (Cambridge University Press, New York, 1960), p. 253.

All 4 components of the $\bar{h}_{\alpha\beta}$ may be eliminated through the use of the coordinate condition $\bar{h}_{\mu\nu,\nu}=0$, and the conversion of spatial derivatives to time derivatives multiplied by direction vectors. This yields

$$\int \frac{dE}{dt}dt=-\frac{1}{32\pi G}\int dt\int_{\text{sphere}} dS[\tfrac{1}{2}(n_in_j\bar{h}_{ij,4})^2 -2n_jn_k\bar{h}_{ij,4}\bar{h}_{ik,4}+\bar{h}_{ij,4}\bar{h}_{ij,4} +n_in_j\bar{h}_{ij,4}\bar{h}_{kk,4}-\tfrac{1}{2}(\bar{h}_{kk,4})^2]. \quad (3.15)$$

Let us examine the quantity within the brackets. If we evaluate it in a system of coordinates where $n_3=1$, $n_1=n_2=0$, then we can write

$$\int \frac{d^2E}{dtdS}dt=-\frac{1}{32\pi G} \times\int dt[\tfrac{1}{2}(\bar{h}_{11,4}-\bar{h}_{22,4})^2+2(\bar{h}_{12,4})^2]. \quad (3.16)$$

Therefore, the radiation of gravitational waves always yields a decrease in energy of the system. This result is valid for any system, relativistic or nonrelativistic. One might object that the result depends on the particular choice of coordinate condition $\bar{h}_{\mu\nu,\nu}=0$. However, it has been shown[10] that if the potentials decrease like $1/r$ for large r, consistency of the field equations requires that (3.16) be true, and therefore also requires that the system lose energy as a result of the radiation.

If we consider a nonrelativistic system again, the $\bar{h}_{ij,4}$ can be found in terms of the Q_{ij} given before.

$$\bar{h}_{ij,4}=-\frac{4G}{r}\left[\frac{d^3Q_{ij}}{dt^3}\right]_{t-r}. \quad (3.17)$$

From (3.15) we get the following angular distribution of the radiation:

$$\int \frac{d^2E}{dtd\Omega}dt=-\frac{1}{8\pi}\int dt\left[\frac{1}{2}\left(n_in_j\frac{d^3Q_{ij}}{dt^3}\right)^2-2n_jn_k\frac{d^3Q_{ik}}{dt^3} \times\frac{d^3Q_{ij}}{dt^3}+\frac{d^3Q_{ij}}{dt^3}\frac{d^3Q_{ij}}{dt^3}+n_in_j\frac{d^3Q_{ij}}{dt^3} \times\frac{d^3Q_{kk}}{dt^3}-\frac{1}{2}\left(\frac{d^3Q_{ii}}{dt^3}\right)^2\right]. \quad (3.18)$$

The only angular dependence in (3.18) is in the n_i. The integral over angles becomes trivial, and one again obtains Eq. (3.13) as the total average energy loss.

IV. ANGULAR MOMENTUM LOSS

The loss of angular momentum of the system can be found by methods analogous to those used in the energy loss case. The details, however, are somewhat more complicated. Starting from (2.10), if we consider only systems where there is no matter entering or leaving the system, then the angular momentum loss must be given by

$$\frac{dL_i}{dt}=-\epsilon_{ijk}\int x_jX_{km}dS_m. \quad (4.1)$$

The argument that X_{km} can be replaced by $X_{km}^{(2)}$ as in the energy radiation case does not hold here. This is because there is the extra factor of x_j in the surface integral. At first glance, it appears that the surface integral is proportional to r for large r and thus diverges. If one examines that part of $X_{km}^{(2)}$ which can give such a dependence, one finds that, over a time average, $X_{km}^{(2)}$ is given by

$$\int X_{km}^{(2)}dt=\frac{1}{32\pi G}\int h_{\alpha\beta,k}\bar{h}_{\alpha\beta,m}dt.$$

Its contribution to (4.1) over a time average will therefore be

$$-\frac{1}{32\pi G}\epsilon_{ijk}\int dt\int dS_m\frac{x_jx_mx_k}{r^2}h_{\alpha\beta,4}\bar{h}_{\alpha\beta,4},$$

which vanishes, since ϵ_{ijk} is antisymmetric in any two indices. Thus we can have a contribution only from the $1/r^3$ part of $X_{km}^{(2)}$ and the $1/r^3$ part of $X_{km}^{(3)}$.

Of all of the terms of $X_{km}^{(3)}$, we get a nonvanishing contribution to the angular momentum loss only from the following term:

$$\int \frac{dL_i'}{dt}dt\equiv\frac{\epsilon_{ijk}}{32\pi G}\int dt\int dS_mx_j\bar{h}_{\alpha\beta}\bar{h}_{k\gamma,\alpha}h_{m\gamma,\beta}. \quad (4.2)$$

We can get an estimate of the order of magnitude of this term in the general case, and in the nonrelativistic case an explicit expression can be obtained in terms of the Q_{ij}. Letting dL_i/dt be the angular momentum loss which one obtains from the terms of $X_{km}^{(2)}$, we have that $dL_i'/dt\approx(GM\omega/c^3)dL_i/dt$, where ω is a characteristic frequency of the system. Therefore, providing that $h_{\mu\nu}\ll 1$, the contribution of (4.2) will be negligible compared with those from $X_{km}^{(2)}$. For a localized nonrelativistic system some cancellation occurs; the terms from $X_{km}^{(3)}$ then contribute in order $(v/c)^5$ times those from $X_{km}^{(2)}$ for a system which is gravitationally bound. For a nongravitationally bound system, they are of order $(GM/ac^2)(v/c)^3$ smaller, where a is a characteristic distance of the system. Thus, we are justified in dropping the terms originating from the third-order field stresses. Equation (4.1) can then be written over a time average as

$$\int \frac{dL_i}{dt}dt=-\epsilon_{ijk}\int dt\int dS_mx_jX_{km}^{(2)}, \quad (4.3)$$

where the $1/r^3$ part of $X_{km}^{(2)}$ is needed.

As in the energy radiation case, we can drop some terms in $X_{km}^{(2)}$ immediately. For example, any term which has a factor $\bar{h}_{\alpha\beta,\lambda\lambda}$ will not contribute to the angular momentum loss in the order in which we are calculating. Since $\bar{h}_{\alpha\beta,\lambda\lambda}\propto h^2$, this type of term will be of order h^3, and by the same argument as used in eliminating the contribution of the third-order stresses, it will give a negligible contribution. Also, any terms in $X_{km}^{(2)}$ which are proportional to δ_{km} can be neglected, since we would then have the integrand in (4.3) symmetrical in j and k, and the indicated sum would vanish. The remaining terms which can contribute are

$$\begin{aligned}X_{km}^{(2)}=(32\pi G)^{-1}\{&-\bar{h}_{\alpha\beta,k}\bar{h}_{\alpha\beta,m}-2\bar{h}_{\gamma k,\delta}\bar{h}_{\gamma m,\delta}\\&+2\bar{h}_{mk,\delta}\bar{h}_{\sigma\sigma,\delta}+2\bar{h}_{k\gamma,\delta}\bar{h}_{m\delta,\gamma}-\bar{h}_{k\gamma,m}\bar{h}_{\sigma\sigma,\gamma}\\&-\bar{h}_{\sigma\sigma,\delta}\bar{h}_{m\delta,k}-\tfrac{1}{2}\bar{h}_{\sigma\sigma,m}\bar{h}_{\gamma\gamma,k}-2\bar{h}_{km,\alpha\beta}\bar{h}_{\alpha\beta}\\&-2\bar{h}_{\alpha\beta}\bar{h}_{\alpha\beta,mk}-\bar{h}_{k\beta}\bar{h}_{\sigma\sigma,\beta m}-\bar{h}_{m\beta}\bar{h}_{\sigma\sigma,\beta k}\\&+2\bar{h}_{\alpha\beta}\bar{h}_{k\alpha,m\beta}+2\bar{h}_{\alpha\beta}\bar{h}_{m\alpha,k\beta}+\bar{h}_{\sigma\sigma}\bar{h}_{\delta\delta,km}\}\,.\end{aligned}\quad(4.4)$$

In each term of (4.4) we have the product of two h's. In order that the product be proportional to $1/r^3$, one of the h's must be proportional to $1/r^2$ and the other proportional to $1/r$. In order to reduce the terms in (4.4) to only those terms which contribute to (4.3), we need one property of the $1/r^2$ part of the potentials and their derivatives. If we have only the $1/r^2$ part of $\bar{h}_{\mu\nu,\lambda}$, denoted by $\bar{h}_{\mu\nu,\lambda}^{(2)}$, and differentiate this again with respect to x_j, we must have that for large r

$$(\bar{h}_{\mu\nu,\lambda}^{(2)})_{,j}=-n_j\bar{h}_{\mu\nu,\lambda 4}^{(2)}\,,\quad(4.5)$$

where terms of order $1/r^3$ have been dropped. This is easily seen from an examination of the retarded solution for $\bar{h}_{\mu\nu,\lambda}$.

Let us suppose that we have a product of two h's, say $\bar{h}_{\alpha\beta,\lambda}\bar{h}_{\sigma\epsilon,\delta}$, and that we wish to consider only the $1/r^3$ part, which is again denoted by a superscript (3). Then, if we differentiate this product with respect to x_i, we must have by the same reasoning that

$$\frac{\partial}{\partial x_i}[\bar{h}_{\alpha\beta,\lambda}\bar{h}_{\sigma\epsilon,\delta}]^{(3)}=-n_i\frac{\partial}{\partial t}[\bar{h}_{\alpha\beta,\lambda}\bar{h}_{\sigma\epsilon,\delta}]^{(3)}\,,$$

where terms of order higher than $1/r^3$ have been neglected. This, together with the fact that terms with a $\Box$ operating on them and terms proportional to δ_{km} yield no contribution to (4.3), allows us to eliminate many terms of (4.4) when substituted into (4.3), and to simplify the others. For example, if we consider the second term of (4.4), we get a contribution to (4.3) proportional to

$$\begin{aligned}&\int dt\int dS_m x_j\bar{h}_{\gamma k,\delta}\bar{h}_{\gamma m,\delta}\\&=-\frac{1}{2}\int dt\int dS_m x_j\bar{h}_{\gamma k,\delta\delta}\bar{h}_{\gamma m}-\frac{1}{2}\int dt\int dS_m x_j\bar{h}_{\gamma k}\bar{h}_{\gamma m,\delta\delta}\\&\quad-\frac{1}{2}\int dt\int dS_m x_j\frac{\partial}{\partial x_p}(\bar{h}_{\gamma k,p}\bar{h}_{\gamma m}+\bar{h}_{\gamma k}\bar{h}_{\gamma m,p})\,.\end{aligned}$$

The first two terms are neglected because of the $,\delta\delta$. In the last term, if the part of the expression within the parentheses is proportional to $1/r^3$, then $\partial/\partial x_p=-n_p(\partial/\partial t)$, and the integral over time vanishes. Therefore, the only part of the quantity which counts is the part proportional to $1/r^2$, or thus where each term is proportional to $1/r$. But then,

$$\bar{h}_{\gamma k,p}\bar{h}_{\gamma m}+\bar{h}_{\gamma k}\bar{h}_{\gamma m,p}=-n_p(\partial/\partial t)(\bar{h}_{k\gamma}\bar{h}_{m\gamma})\,,$$

and the time integral eliminates it.

By similar arguments, each term of (4.4) can be reduced or eliminated when substituted into (4.3). The result of this procedure is that Eq. (4.3) can be written

$$\int\frac{dL_i}{dt}dt=-\epsilon_{ijk}(32\pi G)^{-1}\int dt\int dS_m x_j[h_{\alpha\beta,k}\bar{h}_{\alpha\beta,m}-2\bar{h}_{\alpha\beta,k}\bar{h}_{m\alpha,\beta}-2\bar{h}_{\alpha\beta,m}\bar{h}_{k\alpha,\beta}]\,.\quad(4.6)$$

This is the angular momentum analog of Eq. (3.7).

The angular momentum loss can be found by two methods analogous to those of the energy loss calculation. We may find the radiation reaction loss by converting (4.6) back to a volume integral, or we may integrate (4.6) directly over angles in the nonrelativistic approximation. Carrying out the former, we get, after some simplification,

$$\int\frac{dL_i}{dt}dt=-\tfrac{1}{2}\epsilon_{ijk}\int dt\int dV x_j(h_{\alpha\beta,k}-2\bar{h}_{k\alpha,\beta})S_{\alpha\beta}\,,\quad(4.7)$$

which is the angular momentum analog of the radiation reaction for the energy loss case, Eq. (3.9). Equation (4.7) can be further simplified by an integration by parts to give

$$\int\frac{dL_i}{dt}dt=-\tfrac{1}{2}\epsilon_{ijk}\int dt\int dV(x_jh_{\alpha\beta,k}S_{\alpha\beta}-2h_{\alpha k}S_{\alpha j})\,.\quad(4.8)$$

Using the expansion of the potentials $h_{\mu\nu}$ about the present time, we obtain an expression for (4.8) which is the analog of Eq. (3.11). By the same methods as were used in the energy case, this can be easily reduced over a time average to

$$\int\frac{dL_i}{dt}dt=-\tfrac{2}{5}G\epsilon_{ijk}\int dt\frac{d^2Q_{jm}}{dt^2}\frac{d^3Q_{km}}{dt^3}\,.\quad(4.9)$$

This expression for the ith component of the angular momentum radiated[13] is the analog of the energy radiation equation (3.13).

[13] This result has also been obtained by T. A. Morgan and A. Peres, Phys. Rev. **131**, 494 (1963). However, their derivation again depends on the previous paper of Peres, and is thus not valid for gravitationally bound systems. Also, it depends on the somewhat fortunate choice of the total angular momentum of the system, $\epsilon_{ijk}\int x_jT^{4k}(-g)^{1/2}dV$. Equally valid with their reasoning would be the expression $\epsilon_{ijk}\int x_jT_k^{4}(-g)^{1/2}dV$, where $T_k^4=g_{k\alpha}T^{4\alpha}$, since they assume that $h_{\mu\nu}\ll 1$. This choice would have given a different expression for the angular momentum loss, even in the case of nongravitationally bound systems.

We may also get to (4.9) by way of the calculation of the angular momentum flux crossing a large sphere. This allows us to get the angular momentum radiated as a function of the angles as well as the total radiation. For this we need $\bar{h}_{\mu\nu}$ to order $1/r^2$. This is determined in the following manner. By explicit differentiation of (3.4), one obtains, to order $1/r^2$,

$$\bar{h}_{\mu\nu,k}=-\frac{x_k}{r}\bar{h}_{\mu\nu,4}-\frac{x_k}{r^2}\bar{h}_{\mu\nu}-\frac{4G}{r^2}\int\int \dot{S}_{\mu\nu}x_k'\delta(\ )dt'dV' +\frac{4G}{r^3}x_k\int \dot{S}_{\mu\nu}\mathbf{r}\cdot\mathbf{r}'\delta(\ )dt'dV', \quad (4.10)$$

where $\delta(\ )\equiv\delta(t'-t+|\mathbf{r}-\mathbf{r}'|)$. Let us first consider the case where $(\mu,\nu)=(i,j)$. Then the last two terms in (4.10) will be much smaller than the first in the order of the quadrupole approximation. To see this, we let everything have a time dependence $e^{-i\omega t}$. Then we find that

$$\bar{h}_{ij}(\mathbf{r},\omega)\propto\int S_{ij}(\mathbf{r}',\omega)e^{-i\omega(\mathbf{r}\cdot\mathbf{r}')/r}dV',$$

whereas the last two terms of (4.10) for $(\mu,\nu)=(i,j)$ have the form

$$\int S_{ij}(\mathbf{r}',\omega)[-i\omega x_k']e^{-i\omega(\mathbf{r}\cdot\mathbf{r}')/r}dV'.$$

In the quadrupole approximation, we let $\omega x_k'\ll 1$, so that the $\exp(-i\omega(\mathbf{r}\cdot\mathbf{r}')/r)$ can be set equal to one. The latter two terms have the form of the second term in the expansion of the exponential, and thus, in this approximation, can be neglected. Therefore, to order $1/r^2$, we can write

$$\bar{h}_{ij,k}=-(n_k/r)\bar{h}_{ij}-n_k\bar{h}_{ij,4}, \quad (4.11)$$

where $\bar{h}_{ij}$ is then given by

$$\bar{h}_{ij}=-\frac{4G}{r}\left[\int S_{ij}dV\right]_{t-r}.$$

Let us now consider $\bar{h}_{4m,k}$, setting (μ,ν) equal to $(4,m)$ in (4.10). In the last two terms of (4.10), we can let $\dot{S}_{4m}=S_{mj,j}$, in which case an integration by parts allows us to obtain

$$\bar{h}_{4m,k}=-\frac{n_k}{r}\bar{h}_{4m}-n_k\bar{h}_{4m,4}-\frac{1}{r}\bar{h}_{mk}+\frac{n_mn_j}{r}\bar{h}_{kj}. \quad (4.12)$$

Similarly,

$$\bar{h}_{44,k}=-\frac{n_k}{r}\bar{h}_{44}-n_k\bar{h}_{44,4}-\frac{n_kn_mn_s}{r}\bar{h}_{ms} +\frac{n_kn_m}{r}\bar{h}_{4m}-\frac{1}{r}\bar{h}_{4k}+\frac{n_m}{r}\bar{h}_{mk}. \quad (4.13)$$

The coordinate conditions $\bar{h}_{44,4}=\bar{h}_{4m,m}$ and $\bar{h}_{k4,4}=\bar{h}_{km,m}$ are used to eliminate the time derivative terms from (4.12) and (4.13).

We are now in a position to solve (4.6). In the first term of (4.6), $h_{\alpha\beta,k}$ must be of order $1/r^2$, since any term proportional to x_k will yield zero upon summation with $\epsilon_{ijk}x_j$. Then $\bar{h}_{\alpha\beta,m}=-n_m\bar{h}_{\alpha\beta,4}$. From Eqs. (4.11)–(4.13), we have that

$$\begin{aligned}\epsilon_{ijk}x_j\bar{h}_{44,k4}&=2\epsilon_{ijk}n_jn_m\bar{h}_{mk,4},\\ \epsilon_{ijk}x_j\bar{h}_{4m,k}&=-\epsilon_{ijk}n_j\bar{h}_{mk},\\ \epsilon_{ijk}x_j\bar{h}_{pq,k}&=0,\end{aligned} \quad (4.14)$$

so that for the first term of (4.6) we get

$$+\epsilon_{ijk}(32\pi G)^{-1}\int dt\int_{\text{sphere}}dS[n_qn_jn_mn_p\bar{h}_{qk}\bar{h}_{mp,4} -2n_jn_m\bar{h}_{qk}\bar{h}_{qm,4}+2n_jn_q\bar{h}_{qk}\bar{h}_{pp,4}].$$

The second term of (4.6) yields in an analogous manner

$$-2\epsilon_{ijk}(32\pi G)^{-1}\int dt\int_{\text{sphere}}dS[n_qn_jn_mn_p\bar{h}_{pk}\bar{h}_{qm,4} -n_qn_j\bar{h}_{mk}\bar{h}_{mq,4}].$$

The third term offers somewhat more difficulty. However, straightforward analysis shows that it can be reduced to

$$-2\epsilon_{ijk}\int dt\int_{\text{sphere}}dS[4n_jn_pn_qn_m\bar{h}_{pm,4}\bar{h}_{kq} -n_jn_q\bar{h}_{mm,4}\bar{h}_{kq}-3n_jn_p\bar{h}_{pm,4}\bar{h}_{km}].$$

Therefore, the total angular momentum radiation distribution is given by

$$\int\frac{d^2L_i}{dtd\Omega}dt=\frac{\epsilon_{ijk}}{8\pi}\int dt\left[6n_jn_p\frac{d^2Q_{mk}}{dt^2}\frac{d^3Q_{mp}}{dt^3} -9n_jn_mn_pn_q\frac{d^2Q_{mk}}{dt^2}\frac{d^3Q_{pq}}{dt^3} +4n_jn_m\frac{d^2Q_{mk}}{dt^2}\frac{d^3Q_{pp}}{dt^3}\right], \quad (4.15)$$

where the solutions for $\bar{h}_{ij}$ in terms of the Q_{ij} have been used. The integral over angles is trivially performed. This yields

$$\int\frac{dL_i}{dt}dt=-\tfrac{2}{5}\epsilon_{ijk}\int dt\frac{d^2Q_{mj}}{dt^2}\frac{d^3Q_{mk}}{dt^3}, \quad (4.16)$$

which, of course, agrees with Eq. (4.9).

For a rotating rigid system, say a spinning rod, there is only one parameter ω which specifies the state of the system. We have, however, two equations, dL_i/dt and

dE/dt, to specify the secular change in ω over one period. It is easy to show that the two expressions give the same secular change in ω and are thus consistent. In the case where the motion is not circular, the two equations give us different information. For example, in the case of two point masses moving in elliptical orbits, we can predict not only the secular change in the energy or semimajor axis, but also the secular change in the eccentricity as well. This analysis will be carried out in the next section.

V. SECULAR CHANGES IN THE TWO-POINT MASS SYSTEM

The results of the previous sections can be applied to find the secular change in the elements of the relative orbit of two point masses resulting from gravitational radiation. The equation of the relative orbit of the motion[14] is

$$r=a(1-e^2)/(1+e\cos\psi). \tag{5.1}$$

If the plane of the motion and position of the orbit in the plane is specified, then there are two parameters necessary to describe the orbit: the major axis a and the eccentricity e. In the Newtonian theory, they are constants of the motion. In the general theory of gravitation, they will be functions of time, which will be slowly varying in the nonrelativistic limit. These parameters are related to the total energy E and the relative angular momentum L through the following equations:

$$a=-Gm_1m_2/2E, \tag{5.2}$$

$$L^2=Gm_1^2m_2^2(m_1+m_2)^{-1}a(1-e^2). \tag{5.3}$$

In a previous paper,[15] the energy radiated from this system by gravitational waves was studied in detail. It was found that the time average of the energy emission rate is given by

$$\left\langle\frac{dE}{dt}\right\rangle=-\frac{32}{5}\frac{G^4m_1^2m_2^2(m_1+m_2)}{c^5a^5(1-e^2)^{7/2}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right). \tag{5.4}$$

Applying the analysis of Sec. IV, one finds that the average angular momentum emission rate is given by

$$\left\langle\frac{dL}{dt}\right\rangle=-\frac{32}{5}\frac{G^{7/2}m_1^2m_2^2(m_1+m_2)^{1/2}}{c^5a^{7/2}(1-e^2)^2}(1+\tfrac{7}{8}e^2). \tag{5.5}$$

The equations for $\langle da/dt\rangle$ and $\langle de/dt\rangle$ are derived from (5.4) and (5.5):

$$\left\langle\frac{da}{dt}\right\rangle=-\frac{64}{5}\frac{G^3m_1m_2(m_1+m_2)}{c^5a^3(1-e^2)^{7/2}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right), \tag{5.6}$$

[14] For an elliptical orbit, a denotes the semimajor axis, e the eccentricity, and ψ the angular coordinate in the plane of the orbit.

[15] P. C. Peters and J. Mathews, Phys. Rev. 131, 435 (1963).

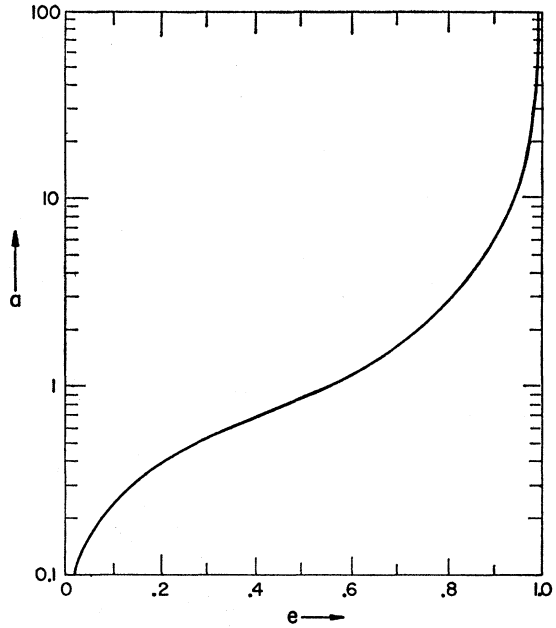

FIG. 1. The semimajor axis a as a function of the eccentricity e in the decay of a two-point mass system. Here, c_0 is chosen to be 1.

$$\left\langle\frac{de}{dt}\right\rangle=-\frac{304}{15}e\frac{G^3m_1m_2(m_1+m_2)}{c^5a^4(1-e^2)^{5/2}}\left(1+\frac{121}{304}e^2\right). \tag{5.7}$$

Hence, during a decay of the orbit for which gravitational radiation is the only energy loss mechanism, we can obtain the differential equation relating a to e during the decay:

$$\left\langle\frac{da}{de}\right\rangle=\frac{12}{19}\frac{a}{e}\frac{[1+(73/24)e^2+(37/96)e^4]}{(1-e^2)[1+(121/304)e^2]}. \tag{5.8}$$

The above equations are sufficient to determine the decay uniquely. Starting from a given orbit with parameters a_0 and e_0, Eqs. (5.6) and (5.7) in principle give enough information to find $a(t)$ and $e(t)$. If we set $e_0=0$ to find the decay of the circular orbit, either (5.6) or (5.7) gives

$$a(t)=(a_0^4-4\beta t)^{1/4}, \tag{5.9}$$

where

$$\beta=\frac{64}{5}\frac{G^3m_1m_2(m_1+m_2)}{c^5}.$$

Therefore, the system decays in a finite time T_c given by

$$T_c(a_0)=a_0^4/(4\beta). \tag{5.10}$$

Consider the case of circularly orbiting binary stars for which we neglect deformation, mass flow, and other radiation processes. We may predict the lifetime of the system for collapse as a result of the radiation of gravitational waves. For binary star systems in which each component has a mass equal to one solar mass, we

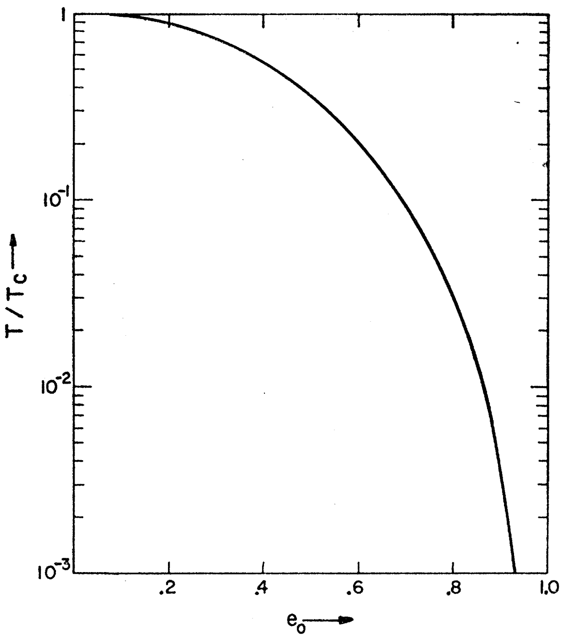

FIG. 2. The ratio of the lifetime of an eccentric system to that of a circular one plotted against the initial eccentricity. This ratio is independent of the initial value of the semimajor axis.

obtain the following lifetimes. For a separation of ~ 10 solar radii, the period is ~ 4.5 days, and the lifetime for decay is $\sim 3\times 10^{12}$ years. For two white dwarfs (radii $\sim 10^9$ cm) separated by 10^{10} cm, the period is ~ 0.0045 days and the lifetime $\sim 3\times 10^4$ years. For the extreme case in which the same two stars are just touching, the lifetime becomes only 50 years.

In order to get the decay time for an $e_0\neq 0$, we must solve the pair of equations (5.6) and (5.7). First we can find $a(e)$ for the decay from (5.8). The integration of this equation is tedious but straightforward. We find $a(e)$ to be

$$a(e)=\frac{c_0e^{12/19}}{(1-e^2)}\left[1+\frac{121}{304}e^2\right]^{870/2299}, \tag{5.11}$$

where c_0 is determined by the initial condition $a=a_0$ when $e=e_0$. Figure 1 displays $a(e)$ versus e. For small e, this reduces to

$$a(e)\approx c_0e^{12/19},\quad e^2\ll 1,$$

and for e near 1, this becomes

$$a(e)\approx c_1/(1-e^2),\quad (1-e^2)\ll 1,$$

where $c_1=c_0(425/304)^{870/2299}\approx 1.1352c_0$. Thus, for all practical purposes, one might neglect the complicated factor and just consider $a(e)$ to be given by

$$a(e)\approx c_0e^{12/19}/(1-e^2). \tag{5.12}$$

From (5.11) and (5.7) we can write the equation giving the time decay of an eccentric system exactly. Since $e\to 0$ as $a\to 0$, $e(t)$ may be considered rather than $a(t)$ in finding $T(a_0,e_0)$, the decay lifetime of the system:

$$\left\langle\frac{de}{dt}\right\rangle=-\frac{19}{12}\frac{\beta}{c_0^4}\frac{e^{-29/19}(1-e^2)^{3/2}}{[1+(121/304)e^2]^{1181/2299}}. \tag{5.13}$$

The lifetime $T(a_0,e_0)$ is then given by the integral

$$T(a_0,e_0)=\frac{12}{19}\frac{c_0^4}{\beta}\times\int_0^{e_0}\frac{de\,e^{29/19}[1+(121/304)e^2]^{1181/2299}}{(1-e^2)^{3/2}}. \tag{5.14}$$

For small e_0, we get

$$T(a_0,e_0)\approx\frac{12}{19}\frac{c_0^4}{\beta}\int_0^{e_0}de\,e^{29/19}=\frac{c_0^4}{4\beta}e_0^{48/19}.$$

This is approximately equal to $a_0^4/4\beta$, agreeing with the lifetime found for the circular case, Eq. (5.10). For e_0 near 1, the lifetime becomes

$$T(a_0,e_0)\approx(768/425)T_c(a_0)(1-e_0^2)^{7/2}.$$

The solution for arbitrary e_0 can be obtained by numerical integration, whose results for $T(a_0,e_0)/T_c(a_0)$ are plotted against e_0 in Fig. 2. One can easily see that for a given initial major axis, the time of collapse decreases rapidly as $e_0\to 1$. This should not be surprising since, for fixed a, dE/dt is proportional to $(1-e^2)^{-7/2}$ if e is near 1; and in general, the system spends most of the decay time in a state for which $a\approx a_0$.

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